THE OSCILLATING WING IN A SUBSONIC FLOW.

R 53, Int 2.

Computation Department Mathematical Centre.

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INTRODUCTION

This report R 53, Int 2 is the second of a number of interim reports giving information about computations carried out by the Computation Department of the Mathematical Centre on behalf of the National Aeronautical Research Institute in Amsterdam under contract R 53. The final report R 53 that will be made up eventually will not contain much else than the final results of the computations and, moreover, will be not available for general distribution. As however in the course of the computations a lot of information has to be compiled for internal use, and part of this information may be of some value to others, this compilation will be done in the form of interim reports, that will be made available for limited circulation.

I. Development of $B_m^{(n)}$ in Factorial-Series.

1. The coefficients $B_m^{(n)}$ of the Fourier-expansion of the se_n-solutions of the Mathieu's equation obey the following recurrence relations. (C.f. Mac-Lachlan: Theory and Application of Mathieu-functions.)

$$\left.\begin{aligned} &(a-1+q)\ B_1^{(n)} - q\ B_3^{(n)} = 0 \\ &\left[a-(2r+1)^2\right] B_{2r+1}^{(n)} - q(B_{2r+3}^{(n)} + B_{2r-1}^{(n)}) = 0 \quad (r \geq 1) \end{aligned}\right\} \quad (1.1)$$

if n is odd

and

$$\left.\begin{aligned} &(a-4)\ B_2^{(n)} - q\ B_4^{(n)} = 0 \\ &(a-4r^2)\ B_{2r}^{(n)} - q(B_{2r+2}^{(n)} + B_{2r-2}^{(n)}) = 0 \qquad (r \geq 2) \end{aligned}\right\} \quad (1.1')$$

if n is even.

The first formulae of both (1.1) and (1.1') differ from the general ones, but one can easily see that we can put them in that general form by the introduction of $B_m^{(n)}$'s with non-positive index m. In that way (1.1) becomes

$$\left.\begin{aligned} &\left[a-(2r+1)^2\right] B_{2r+1}^{(n)} - q(B_{2r+3}^{(n)} + B_{2r-1}^{(n)}) = 0 \\ &B_1^{(n)} = - B_{-1}^{(n)}, \end{aligned}\right\} \quad (1.2)$$

and (1.1')

$$\left.\begin{aligned} &(a-4r^2)\ B_{2r}^{(n)} - q\ (B_{2r+2}^{(n)} + B_{2r-2}^{(n)}) = 0 \\ &B_0^{(n)} = 0 \end{aligned}\right\} \quad (1.2')$$

Now it immediately follows, that the $B_m^{(n)}$ should have an antimetrical character for ~~integervalues~~ of r, i.e. in those points that interest us only. Moreover these $B_m^{(n)}$ should get small for large positive values of r. These two conditions can only be satisfied if a has its characteristic value.

We shall now assume that a actually has its characteristic value b_n, and try to solve the $B_m^{(n)}$ by means of expansion in factorial-series, making use of the fact, that we are looking for the small solution. Then the initial conditions are automatically satisfied.

2. Replacing r in (1.2') by $r + \frac{1}{2}$ we get the formulae (1.2), and so we see that the two cases of odd and even values of n are essentially identical. We shall therefore only consider even values of n and translate the obtained results for odd values of n.

The small solution of the difference-equations (1.2') can asymptotically be decribed by $A(-q/4)^r(r!)^{-2}$ for large values of r,

$$B_{2r}^{(n)} = \frac{(-\vartheta)^r}{(r-p)!(r+p)!} \; C_{r,p}^{(n)} \tag{2.1}$$

with $\vartheta = \frac{q}{4}$ and p arbitrary.

As

$$(r-p)!(r+p)! \sim (r!)^2$$

$C_{r,p}^{(n)}$ shall tend to a finite value if $r \to \infty$, that is idependent of p. We expand $C_{r,p}^{(n)}$ in a factorial-series:

$$C_{r,p}^{(n)} = \sum_{\nu=0}^{\infty} \frac{\gamma_{\nu,p}^{(n)}}{(r-p+1)\dots(r-p+\nu)} \tag{2.2}$$

in which $\gamma_{0,p}^{(n)}$ is independent of p, but depends only on the way in which the $B_{2r}^{(n)}$ are normalized.

Then from (1.2') we get, substituting for a its characteristic value b_n:

$$\left(r^2 - \frac{b_n}{4}\right)(r-p+1)(r+p+1)\, C_{r,p}^{(n)} - (r-p)(r-p+1)(r+p)(r+p+1)\, C_{r-1,p}^{(n)} - \vartheta^2 C_{r+1,p}^{(n)} = 0 \tag{2.3}$$

In this

$$C_{r+1,p}^{(n)} = \sum_{\nu=0}^{\infty} \frac{\gamma_{\nu,p}^{(n)}(r-p+1)!}{(r+\nu-p+1)!}, \tag{2.4}$$

and

$$\left(r^2 - \frac{b_n}{4}\right)(r-p+1)(r+p+1)\, C_{r,p}^{(n)} = \left(r^2 - \frac{b_n}{4}\right)(r+p+1) \sum \frac{\gamma_{\nu,p}^{(n)}(r-p+1)!}{(r+\nu-p)!}$$

$$= \sum_{\nu=0}^{\infty} \frac{\delta_{\nu,p}^{(n)}(r-p+1)!}{(r+\nu-p+1)!} \tag{2.5}$$

with

$$\delta_{\nu,p}^{(n)} = \gamma_{\nu+4,p}^{(n)} + a_{\nu+3,p}^{(n)}\gamma_{\nu+3,p}^{(n)} + b_{\nu+2,p}^{(n)}\gamma_{\nu+2,p}^{(n)} + c_{\nu+1,p}^{(n)}\gamma_{\nu+1,p}^{(n)} \tag{2.6}$$

the $a_{\nu,p}^{(n)}$, $b_{\nu,p}^{(n)}$ and $c_{\nu,p}^{(n)}$ being defined by

$$\left(r^2 - \frac{b_n}{4}\right)\ (r+p+1)=(r+\nu-p)(r+\nu-p-1)(r+\nu-p-2)+$$

$$a^{(n)}_{\nu,p}\ (r+\nu-p)(r+\nu-p-1)+ b^{(n)}_{\nu,p}\ (r+\nu-p)+ c^{(n)}_{\nu,p}\ , \qquad (2.7)$$

and finally

$$(r-p)(r-p+1)(r+p)(r+p+1)\ C^{(n)}_{r-1,p}=(r+p+2)(r+p+1)\sum_\nu \frac{\alpha^{(n)}_{\nu,p}(r-p+1)!}{(r+\nu-p-1)!}$$

$$= \sum_\nu \frac{\beta^{(n)}_{\nu,p}(r-p+1)!}{(r+\nu-p+1)!} \qquad (2.8)$$

with

$$\beta^{(n)}_{\nu,p} = \alpha^{(n)}_{\nu+4,p} + g_{\nu+3,p}\,\alpha^{(n)}_{\nu+3,p} + h_{\nu+2,p}\,\alpha^{(n)}_{\nu+2,p}\ , \qquad (2.9)$$

the $g_{\nu,p}$ and $h_{\nu,p}$ following from

$$(r+p)(r+p+1)=(r+\nu-p-1)(r+\nu-p-2)+ g_{\nu,p}(r+\nu-p-1)+ h_{\nu,p} \qquad (2.10)$$

Substituting (2.4), (2.5) and (2.8) into (2.3), and equating the coefficients of the terms with $\frac{(r-p+1)!}{(r+\nu-p+1)!}$ with the same value of ν we get

$$\beta^{(n)}_{\nu,p} - \gamma^{(n)}_{\nu,p} - \tau^2\,\alpha^{(n)}_{\nu,p} = 0$$

or

$$(g_{\nu+3,p} - a^{(n)}_{\nu+3,p})\ \alpha^{(n)}_{\nu+3,p} + (h_{\nu+2,p} - b^{(n)}_{\nu+2,p})\ \alpha^{(n)}_{\nu+2,p} -$$

$$c^{(n)}_{\nu+1,p}\ \alpha^{(n)}_{\nu+1,p} + \tau^2\ \alpha^{(n)}_{\nu,p} = 0, \qquad (2.11)$$

a recurrence-relation, from which we can calculate the α's successively, making use of $\alpha^{(n)}_{\nu,p} = 0$ for $\nu < 0$, and first taking $\alpha^{(n)}_{0,p} = 1$ which gives values of $B^{(n)}_m$ that differ from the ones we want, by a multiplicative constant, that can be determined later on with the aid of the normalization-rule $\sum B^{(n)2}_m = 1$.

From (2.7) and (2.10) we get

$$\left.\begin{aligned} c^{(n)}_{\nu,p} &= -\ (\nu-2p-1)\left[(\nu-p)^2 - \frac{b_n}{4}\right] \\ h_{\nu,p} - b^{(n)}_{\nu,p} &= -\ (\nu-p)^2 - \nu(\nu-2p-2)+ \frac{b_n}{4} \\ g_{\nu,p} - a^{(n)}_{\nu,p} &= \nu\ , \end{aligned}\right\} \qquad (2.12)$$

and so we can write (2.11) im the form, if we introduce $\lambda_n = -{}^{b_n}\!/\!_4$,

$$\nu\gamma^{(n)}_{\nu,p} - \{(\nu-p-1)^2 + (\nu-1)(\nu-2p-3) + \lambda_n\}\gamma^{(n)}_{\nu-1,p} + (\nu-2p-3)\{(\nu-p-2)^2 + \lambda_n\}\gamma^{(n)}_{\nu-2,p} + \tau^2\gamma^{(n)}_{\nu-3,p} = 0 \tag{2.13}$$

We can put this recurrence-formula in a form more suited for computation by the introduction of an auxiliary quantity.

$$\delta^{(n)}_{\nu-3,p} = \tau^2\gamma^{(n)}_{\nu-3,p} - (\nu-2p-3)\left[(\nu-1)\gamma^{(n)}_{\nu-1,p} - \{(\nu-p-2)^2 + \lambda_n\}\gamma^{(n)}_{\nu-2,p}\right] \tag{2.14}$$

Then we have

$$\nu\gamma^{(n)}_{\nu,p} = \{(\nu-p-1)^2 + \lambda_n\}\gamma^{(n)}_{\nu-1,p} - \delta^{(n)}_{\nu-3,p}\,, \tag{2.15a}$$

and from (2.14) and (2.15a)

$$\delta^{(n)}_{\nu-3,p} = \tau^2\gamma^{(n)}_{\nu-3,p} + (\nu-2p-5)\,\delta^{(n)}_{\nu-4,p}\,,$$

or

$$\delta^{(n)}_{\nu,p} = \tau^2\gamma^{(n)}_{\nu,p} + (\nu-2p)\,\delta^{(n)}_{\nu-1,p}\,. \tag{2.15b}$$

3. Now resuming the results we find that for even values of n we have

$$B^{(n)}_{2r} = (-1)^r \frac{\tau^r}{(r+p)!}\sum_{\nu=0}^{\infty}\frac{\gamma^{(n)}_{\nu,p}}{(r-p+\nu)!}\,, \tag{3.1}$$

with

$$\tau = \frac{q}{4}\,, \tag{3.2}$$

and if

$$\lambda_n = \frac{-b_n}{4} \tag{3.3}$$

$$\nu\gamma^{(n)}_{\nu,p} = \{(\nu-p-1)^2 + \lambda_n\}\gamma^{(n)}_{\nu-1,p} - \delta^{(n)}_{\nu-3,p} \tag{3.4}$$

$$\delta^{(n)}_{\nu,p} = \tau^2\gamma^{(n)}_{\nu,p} + (\nu-2p)\,\delta^{(n)}_{\nu-1,p}\,. \tag{3.5}$$

Quite similarly for odd values of n

$$B^{(n)}_{2r+1} = (-1)^r \frac{\tau^r}{(r+p+1)!}\sum_{\nu=0}^{\infty}\frac{\gamma^{(n)}_{\nu,p}}{(r-p+\nu)!} \tag{3.1'}$$

with

$$\tau = \frac{q}{4}\,, \tag{3.2'}$$

and if $\lambda^*_n = \dfrac{1-b_n}{4}$ (3.3')

$$\nu\gamma^{(n)}_{\nu,p} = \{(\nu-p-1)(\nu-p-2)+\lambda^*_n\}\,\gamma^{(n)}_{\nu-1,p} - \gamma^{(n)}_{\nu-3,p} \qquad (3.4')$$

$$\delta^{(n)}_{\nu,p} = \tau^2\,\gamma^{(n)}_{\nu,p} + (\nu-2p-1)\,\delta^{(n)}_{\nu-1,p}. \qquad (3.5')$$

4. As to the convergence of the factorial-series, we shall again consider the even case. Let

$$\gamma^{(n)}_{\nu,p} = \nu!\,\varepsilon^{(n)}_{\nu,p} \qquad (4.1)$$

Substituting this into (2.13) yields

$$\nu^2(\nu-1)\,\varepsilon^{(n)}_{\nu,p} - (\nu-1)\{2\nu^2-(4p+6)\nu+(p+1)^2+2p+3+\lambda_n\}\,\varepsilon^{(n)}_{\nu-1,p}$$
$$+(\nu-2p-3)\{(\nu-p-2)^2+\lambda_n\}\,\varepsilon^{(n)}_{\nu-2,p} + \frac{\tau^2}{\nu-2}\,\varepsilon^{(n)}_{\nu-3,p} = 0 \qquad (4.2)$$

For large values of ν this gives asymptotically

$$\varepsilon^{(n)}_{\nu,p} - \left\{2 - \frac{4p+6}{\nu} + \frac{(p+2)^2+\lambda_n}{\nu^2}\right\}\varepsilon^{(n)}_{\nu-1,p} + \left\{1 - \frac{4p+6}{\nu} + \frac{(p+2)^2+(2p+3)(2p+4)+\lambda_n}{\nu^2}\right\}\varepsilon^{(n)}_{\nu-2,p} = 0 \qquad (4.3)$$

Putting $\varepsilon^{(n)}_{\nu,p} \sim \dfrac{(\nu-k)!}{\nu!}$ gives $k = 2p+3$ and $k = 2p+2$. Therefore, $\gamma^{(n)}_{\nu,p} \sim (\nu-2p-3)!$ and $\gamma^{(n)}_{\nu,p} \sim (\nu-2p-2)!$ gives the asymptotic form of two independent solutions of the difference-equation (2.13). The third solution is a small one. If we suppose for it that $\nu\gamma^{(n)}_{\nu,p} = O(\nu^3\gamma^{(n)}_{\nu-2,p})$ and $\nu^2\gamma^{(n)}_{\nu-1,p} = O(\nu^3\gamma^{(n)}_{\nu-2,p})$, then the asymptotic form of (2.13) becomes

$$(\nu-2p-3)\{(\nu-p-2)^2+\lambda_n\}\,\gamma^{(n)}_{\nu-2,p} + \tau^2\,\gamma^{(n)}_{\nu-3,p} = 0. \qquad (4.4)$$

having a solution with asymptotic behaviour

$$\gamma^{(n)}_{\nu,p} \sim \frac{(-1)^{\nu}\,\tau^{2\nu}}{(\nu-2p-1)!\,\{(\nu-p)!\}^2} \qquad (4.5)$$

The largest solution behaves as $(\nu-2p-2)!$ So we make a safe estimate of the region of convergence of the factorial-series, if we suppose that the $\gamma^{(n)}_{\nu,p}$ we want, contains the large solution with a non-zero-coefficient, therefore

$$\gamma^{(n)}_{\nu,p} \sim A(\nu-2p-2)! \tag{4.6}$$

Then the general term of the factorial-series behaves as

$$\frac{\gamma^{(n)}_{\nu,p}}{(r-p+\nu)!} \sim \frac{A(\nu-2p-2)!}{(\nu+r-p)!} \sim A\,\nu^{-r-p-2} \tag{4.7}$$

Therefore we have (absolute) convergence as long as

$$r > -p-1, \tag{4.8}$$

the convergence being of the same type as that of a hyperharmonic series with exponent $r+p+2$ (supposing $A \neq 0$).

Similarly in the case of odd n

$$\frac{\gamma^{(n)}_{\nu,p}}{(r-p+\nu)!} \sim A\,\nu^{-r-p-3}, \tag{4.7'}$$

and there is convergence as long as

$$r > -p-2, \tag{4.8'}$$

the convergence being of the same rate as in the case of a hyperharmonic series with exponent $r+p+3$.

5. For different values of p we get different factorial-series. We may refer to the general theory of factorial-series as developed by Nörlund (Cf. his work on "Differenzenrechnung") as to the fact that those series define analytical functions within their region on convergence, which are identical there.

In particular we have (for even n again)

$$B^{(n)}_{2r} = \frac{(-1)^r}{(r+p)!} \sum_{\nu=0}^{\infty} \frac{\gamma^{(n)}_{\nu,p}}{(r-p+\nu)!}$$

$$= \frac{(-1)^r}{(r+p+1)!} \sum_{\nu=0}^{\infty} \frac{\gamma^{(n)}_{\nu,p}(r+p+1)}{(r-p+\nu)!}$$

$$= \frac{(-1)^r}{(r+p+1)!} \sum_{\nu=0}^{\infty} \frac{\gamma^{(n)}_{\nu,p}}{(r-p-1+\nu)!} - \frac{\gamma^{(n)}_{\nu,p}(\nu-2p-1)}{(r-p+\nu)!}$$

$$= \frac{(-1)^r}{(r+p+1)!} \sum_{\nu=0}^{\infty} \frac{\gamma^{(n)}_{\nu,p} - \gamma^{(n)}_{\nu-1,p}(\nu-2p-2)}{(r-p-1+\nu)!} \tag{5.1}$$

Therefore,

$$\gamma^{(n)}_{\nu,p+1} = \gamma^{(n)}_{\nu,p} - (\nu-2p-2)\,\gamma^{(n)}_{\nu-1,p}\ . \tag{5.2}$$

Similarly for odd n,

$$\gamma^{(n)}_{\nu,p+1} = \gamma^{(n)}_{\nu,p} - (\nu-2p-3)\,\gamma^{(n)}_{\nu-1,p}\ . \tag{5.2'}$$

6. Another way of expanding the $B^{(n)}_m$ in factorial series we get in the following manner. Again starting from $B^{(n)}_{2r} \quad A(-\tau)^r/(r!)^{-2}$ and using

$$(2r+\tfrac{1}{2})! \sim \frac{2^{2r}}{\sqrt{2\pi}} \quad \{r!\}^2, \text{ we find}$$

$$B^{(n)}_{2r} \sim \frac{A(-q)^r}{\sqrt{2\pi}\,(2r+\frac{1}{2})!} \tag{6.1}$$

Therefore, if we put $B^{(n)}_{2r} = \dfrac{(-q)^r}{2} \cdot \dfrac{D^{(n)}_r}{(2r+\frac{1}{2})!}$, (6.2)

we find

$$\lim_{r\to\infty} D^{(n)}_r = \lim_{r\to\infty} C^{(n)}_{r,p} \tag{6.3}$$

We now expand $D^{(n)}_r$ in a factorial series

$$D^{(n)}_r = \sum_{\nu=0}^{\infty} \frac{\eta^{(n)}_\nu (2r+\frac{1}{2})!}{(2r+\frac{1}{2}+\nu)!} \tag{6.4}$$

It then follows from (6.3) that

$$\eta^{(n)}_o = \gamma^{(n)}_{o,p}\ . \tag{6.5}$$

From (1.2') we get the recurrence-relation

$$-(4r^2-b_n)(2r+\tfrac{5}{2})(2r+\tfrac{3}{2})\,D^{(n)}_r + (2r+\tfrac{5}{2})(2r+\tfrac{3}{2})(2r+\tfrac{1}{2})(2r-\tfrac{1}{2})\,D^{(n)}_{r-1}$$
$$+ q^2\,D^{(n)}_{r+1} = 0 \tag{6.6}$$

In this we have

$$D^{(n)}_{r+1} = \sum_{\nu=0}^{\infty} \frac{\eta^{(n)}_\nu (2r+\frac{5}{2})!}{(2r+\frac{5}{2}+\nu)!} \tag{6.7}$$

$$(4r^2-b_n)(2r+\tfrac{5}{2})(2r+\tfrac{3}{2})\,D^{(n)}_r = \sum_{\nu=0}^{\infty} (4r^2-b_n)\frac{(2r+\frac{5}{2})!}{(2r+\frac{1}{2}+\nu)!}\,\eta^{(n)}_\nu$$

$$= \sum_{\nu=-4}^{\infty} \frac{(2r+\frac{5}{2})!}{(2r+\nu+\frac{5}{2})!} \zeta_{\nu}^{(n)} \tag{6.8}$$

with

$$\zeta_{\nu}^{(n)} = \eta_{\nu+4}^{(n)} - 2(\nu+3)\eta_{\nu+3}^{(n)} + \left\{(\nu+\tfrac{5}{2})^2 - b_n\right\} \eta_{\nu+2}^{(n)} \tag{6.9}$$

as

$$(4r^2-b_n)=(2r+\tfrac{1}{2}+\nu)(2r-\tfrac{1}{2}+\nu)- 2\nu(2r+\tfrac{1}{2}+\nu)+(\nu+\tfrac{1}{2})^2 -b_n. \tag{6.10}$$

Finally,

$$(2r+\tfrac{5}{2})(2r+\tfrac{3}{2})(2r+\tfrac{1}{2})(2r-\tfrac{1}{2})\, D_{r-1}^{(n)} = \sum_{\nu=0}^{\infty} \frac{(2r+\frac{5}{2})!}{(2r-\frac{3}{2}+\nu)!} \eta_{\nu}^{(n)}$$

$$= \sum_{\nu=-4}^{\infty} \frac{(2r+\frac{5}{2})!}{(2r+\frac{5}{2}+\nu)!} \eta_{\nu+4}^{(n)} \tag{6.11}$$

New (6.6) yields

$$- \zeta_{\nu}^{(n)} + \eta_{\nu+4}^{(n)} + q^2 \eta_{\nu}^{(n)} = 0$$

or

$$2\nu\, \eta_{\nu}^{(n)} = \left[\nu(\nu-1)+\mu\right] \eta_{\nu-1}^{(n)} - q^2 \eta_{\nu-3}^{(n)} \tag{6.12}$$

with

$$\mu = \tfrac{1}{4} - b_n . \tag{6.13}$$

For the difference-equation (6.12) we can give independent solutions, with asymptotic behaviour respectively as

$$\eta_{\nu}^{(n)} \sim 2^{-\nu} (\nu-1)! , \tag{6.14}$$

$$\eta_{\nu}^{(n)} \sim \frac{q^{\nu}}{(\nu+1)!} , \tag{6.15}$$

and

$$\eta_{\nu}^{(n)} \sim \frac{(-q)^{\nu}}{(\nu+1)!} . \tag{6.16}$$

Assuming again that the first and largest one enters into the solution we look for with a non-zero-coefficient, we make a safe estimate of the region of convergence of the factorial-series for $B_m^{(n)}$. We then have

$$\eta_{\nu}^{(n)} \sim A.\ 2^{-\nu}(\nu-1)! \tag{6.17}$$

and so in

$$B_{2r}^{(n)} = \frac{(-q)^r}{\sqrt{2\pi}} \sum_{\nu=0}^{\infty} \frac{\eta_{\nu}^{(n)}}{(2r+\frac{1}{2}+\nu)!} \tag{6.18}$$

$$\frac{\eta_{\nu}^{(n)}}{(2r+\frac{1}{2}+\nu)!} \sim A.2^{-\nu} \frac{(\nu-1)!}{(2r+\frac{1}{2}+\nu)!} \sim A.2^{-\nu} \nu^{-2r-\frac{3}{2}} \tag{6.19}$$

We see, that we are sure of convergence for any value of r, the terms going down as $2^{-\nu}\nu^{-2r-\frac{3}{2}}$. The final convergence of the factorial-series now obtained is therefore much better than that of those with the γ-coefficients. It appeared, however, that at the beginning the convergence of the γ-series was much better, if p was suitably chosen. For this reason we preferred the last ones for actual computation. On the other hand we do still mention the η-series, as they will appear to be usefull when slow-converging series involving the $B_m^{(n)}$ should be summed.

For odd values of n, we put

$$B_{2r+1}^{(n)} = (-q)^r \sqrt{\frac{2}{\pi}} \sum_{\nu=0}^{\infty} \frac{\eta_{\nu}^{(n)}}{(2r+\frac{3}{2}+\nu)!} \tag{6.20}$$

Then

$$\eta_{o}^{(n)} = \gamma_{o,p}^{(n)} \tag{6.21}$$

$$2\nu\, \eta_{\nu}^{(n)} = \left[\nu(\nu-1)+\mu\right] \eta_{\nu-1}^{(n)} - q^2\, \eta_{\nu-3}^{(n)} \tag{6.22}$$

and

$$\frac{\eta_{\nu}^{(n)}}{(2r+\frac{3}{2}+\nu)!} \sim A\, 2^{-\nu} \nu^{-2r-\frac{5}{2}} \tag{6.23}$$

7. We shall not give a detailed proof of the fact, that the γ- and η-series also give the same interpolatory functions of the $B_m^{(n)}$, only the relation between the γ- and η-coefficient will be given.

Putting

$$\frac{1}{4^r(r+p)!(r+\nu-p)!} = \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{\infty} \frac{\omega_j(\nu,p)}{(2r+\nu+\frac{1}{2}+j)!} \tag{7.1}$$

into (3.1), one gets

$$B_{2r} = \frac{(-q)^r}{\sqrt{2\pi}} \sum_{\nu=0}^{\infty} \gamma_{\nu,p}^{(n)} \sum_{j=0}^{\infty} \frac{\omega_j(\nu,p)}{(2r+\nu+\frac{1}{2}+j)!}$$

$$= \frac{(-q)^r}{\sqrt{2\pi}} \sum_{h=0}^{\infty} \frac{1}{(2r+h+\frac{1}{2})!} \sum_{\nu=0}^{h} \gamma_{\nu,p}^{(n)}\, \omega_{h-\nu}(\nu,p) \tag{7.2}$$

and therefore

$$\eta_h^{(n)} = \sum_{\nu=0}^{h} \omega_{h-\nu}(\nu,p)\, \gamma_{\nu,p}^{(n)} \tag{7.3}$$

For the ω's we find at first

$$\omega_o(\nu,p) = \sqrt{2\pi} \lim_{r\to\infty} \frac{(2r+\nu+\frac{1}{2})!}{4^r (r+p)!(r+\nu-p)!} = 2^{\nu} \tag{7.4}$$

and the recurrence-relation

$$2\, j\, \omega_j(\nu,p) = (j-\tfrac{1}{2}-2p+\nu)(j-\tfrac{1}{2}+2p-\nu)\,\omega_{j-1}(\nu,p) \tag{7.5}$$

This yields

$$\omega_j(\nu,p) = 2^{\nu-j} \frac{(j-\frac{1}{2}-2p+\nu)!\,(j-\frac{1}{2}+2p-\nu)!}{j!\,(-\frac{1}{2}-2p+\nu)!\,(-\frac{1}{2}+2p-\nu)!} \tag{7.6}$$

Equally for odd n

$$\eta_h^{(n)} = \sum_{\nu=0}^{h} \omega_{h-\nu}(\nu,p)\, \gamma_{\nu,p}^{(n)} \tag{7.3'}$$

with

$$\omega_o(\nu,p) = 2^{\nu} \tag{7.4'}$$

$$2\, j\, \omega_j(\nu,p) = (j-\tfrac{3}{2}-2p+\nu)(j+\tfrac{1}{2}+2p-\nu)\,\omega_{j-1}(\nu,p) \tag{7.5'}$$

$$\omega_j(\nu,p) = 2^{\nu-j} \frac{(j-\frac{3}{2}-2p+\nu)!\,(j+\frac{1}{2}+2p-\nu)!}{j!\,(-\frac{3}{2}-2p+\nu)!\,(\frac{1}{2}+2p-\nu)!} \tag{7.6'}$$

We only give the relations between η and γ forsake of completeness, as a matter of fact one can better derive the values of directly from their recurrence-relation.

II. Development of the Fourier-coefficients $g_m^{(n)}$ of the non-periodical solution.

8. We define a non-periodic solution $ge_n(z)$ in the same way as Mac-Lachlan, only the normalization is done in that way that we put the non-periodic part exactly $+z\ se_n(z)$ so we have

$$ge_n(z) = +\ z\ se_n(z) + \sum_{m=0}^{\infty} g_m^{(n)} \cos m z \tag{8.1}$$

and correspondingly for the modified functions

$$Ge_n(z) = -\ z\ Se_n(z) + \sum_{m=0}^{\infty} g_m^{(n)} \cosh m z. \tag{8.2}$$

For the $g_m^{(n)}$ exist the recurrence-relations (Cf. Mac-Lachlan)

$$(b_n - 1 - q)\ g_1^{(n)} - q\ g_3^{(n)} = -\ 2\ B_1^{(n)}$$

$$\left[b_n - (2r+1)^2\right] g_{2r+1}^{(n)} - q(g_{2r+3}^{(n)} + g_{2r-1}^{(n)}) = -\ 2(2r+1) B_{2r+1}^{(n)} \qquad (r \geqslant 1) \tag{8.3}$$

for odd values of n

and

$$b_n\, g_0^{(n)} - q\ g_2^{(n)} = 0$$

$$(b_n - 4) g_2^{(n)} - q(g_4^{(n)} + 2\ g_0^{(n)}) = -\ 4\ B_2^{(n)} \tag{8.3'}$$

$$(b_n - 4r^2) g_{2r}^{(n)} - q(g_{2r+2}^{(n)} + g_{2r-2}^{(n)}) = -\ 4r\ B_{2r}^{(n)} \qquad (r \geqslant 2)$$

for even values of n.

The first equation of (8.3) can again be fit into the general form by the introduction of non-positive values of m and the condition $g_{-1}^{(n)} = g_1^{(n)}$, which implies that the $g_{2r+1}^{(n)}$'s should have a symmetrical character for integer values of r.

The first two equations of (8.3') can be fit into the general form in a somewhat more complicated manner. Not $g_0^{(n)}$ should be the value of the function that obeys the general recurrence-relation in the point r=0, but $2\ g_0^{(n)}$, and further there is the condition $g_{-2}^{(n)} = g_2^{(n)}$, also implying a symmetrical character of $g_{2r}^{(n)}$ for integer values of r. Therefore in this case, we shall denote a function that obeys the general recurrence-relation and assume the values $g_{2r}^{(n)}$ for r> 0 by $\underline{g}_{2r}^{(n)}$, then we have

$$g_0^{(n)} = \tfrac{1}{2}\,\underline{g}_0^{(n)} \tag{8.4}$$

and from the first equation of (8.3')

$$b_n\,\underline{g}_0^{(n)} = 2\,q\,\underline{g}_2^{(n)} \tag{8.5}$$

9. If $B_m^{(n)}$ can be interpolated by an analytical function obeying the recurrence-relation of the B, one easily sees that that $-\dfrac{\partial B_m^{(n)}}{\partial m} = -\tfrac{1}{2}\dfrac{\partial B_m^{(n)}}{\partial r}$ is a particular solution of the recurrence-relation of the $g_m^{(n)}$, that becomes small for large values of r. The general small solution of the recurrence-relation is therefore, with α independent of m,

$$g_m^{(n)} = -\tfrac{1}{2}\,\frac{\partial B_m^{(n)}}{\partial r} + \alpha\,B_m^{(n)} \tag{9.1}$$

as the recurrence-relations are the same as for $B_m^{(n)}$ but for the inhomogeneous term.

Therefore

$$T_m^{(n)} = -\tfrac{1}{2}\,(-\tau)^r\,\frac{\partial}{\partial r}\left\{(-\tau)^{-r}\,B_m^{(n)}\right\} \tag{9.2}$$

is a solution, because

$$-\tfrac{1}{2}\,(-\tau)^r\,\frac{\partial}{\partial r}\left\{(-\tau)^{-r}\,B_m^{(n)}\right\} =$$
$$-\tfrac{1}{2}\,\frac{\partial}{\partial r}\,B_m^{(n)} + \tfrac{1}{2}\log(-\tau)B_m^{(n)}$$

Defining $B_m^{(n)}$ by one of the factorial-series of the preceding chapter one can also define $T_m^{(n)}$ according to (9.2).

Now one has

$$\underline{g}_m^{(n)} = T_m^{(n)} + \mathcal{g}^{(n)}\,B_m^{(n)} \tag{9.3}$$

$\mathcal{g}^{(n)}$ can be calculated from the first relation of (8.3) or from (8.5), according to whether n is odd or even.

For odd n one has

$$(b_n-1-q)\,T_1^{(n)} - q\,T_3^{(n)} + \mathcal{g}^{(n)}\left\{(b_n-1-q)B_1^{(n)} - q\,B_3^{(n)}\right\} = - 2\,B_1^{(n)}$$

but according to (1.1)

$$q\,B_3^{(n)} = (b_n-1+q)\,B_1^{(n)},\ \text{so}$$

$$\vartheta^{(n)} = \frac{1}{q} + \frac{(b_n - 1 - q)T_1^{(n)} - q\ T_3^{(n)}}{2\ q\ B_1^{(n)}} \tag{9.4}$$

For even n this becomes

$$b_n\ T_o^{(n)} - 2\ q\ T_2^{(n)} + \vartheta^{(n)} \left\{ b_n\ B_o^{(n)} - 2\ q\ B_2^{(n)} \right\} = 0$$

but $B_o^{(n)} = 0$, so

$$\vartheta^n = \frac{b_n\ T_o^{(n)} - 2q\ T_2^{(n)}}{2\ q\ B_2^{(n)}} \tag{9.5}$$

10. Differentiation of the factorial-series for the $B_m^{(n)}$ easily gives the following developments of the $T_m^{(n)}$

n odd.
$$T_{2r+1}^{(n)} = \frac{(-1)^r \tau^r}{2(r+p+1)!} \sum_{\nu=0}^{\infty} \frac{\gamma_{\nu,p}^{(n)}}{(r-p+\ )!} \left\{ \psi(r-p+\nu) + \psi(r+p+1) \right\} \tag{10.1}$$

the ψ-function being defined as the logarithmical derivative of the factorial-function

$$\text{or } T_{2r+1}^{(n)} = (-q)^r \sqrt{\frac{2}{\pi}} \sum_{\nu=0}^{\infty} \frac{\eta_\nu^{(n)}}{(2r+\frac{3}{2}+\nu)!} \left\{ \psi(2r+\tfrac{3}{2}+\nu) + \log 2 \right\} \tag{10.2}$$

n even
$$T_{2r}^{(n)} = \frac{(-1)^r \tau^r}{2(r+p)!} \sum_{\nu=0}^{\infty} \frac{\gamma_{\nu,p}^{(n)}}{(r-p+\nu)!} \left\{ \psi(r-p+\nu) + \psi(r+p) \right\} \tag{10.3}$$

or
$$T_{2r}^{(n)} = \frac{(-q)^r}{\sqrt{2\pi}} \sum_{\nu=0}^{\infty} \frac{\eta_\nu^{(n)}}{(2r+\frac{1}{2}+\nu)!} \left\{ \psi(2r+\tfrac{1}{2}+\nu) + \log 2 \right\} \tag{10.4}$$

III. Calculation of $Ne_n^{(2)}(z)$.

11. The $Ne_n^{(2)}$-function is defined as that solution of Mathieu's equation that behaves as $\frac{e^{-ike^z}}{\sqrt{ke^z}}$ when $z \to +\infty (k=\sqrt{q})$, or, which is all the same

$$Ne_n^{(2)}(z) \sim A\, H_l^{(2)}(ke^z) \text{ as } z \to +\infty \tag{11.1}$$

for arbitrary l.

Here we do not bother about any normalization, so we can dispense with the exact value of the factor A.

Now we shall try to write $Ne_n^{(2)}(z)$ in the form

$$Ne_n^{(z)}(z) = a\, Se_n(z) + b\, Ge_n(z) \tag{11.2}$$

Only the ratio $\frac{a}{b}$ interests us.

12. Now, first taking n to be odd, we have

$$Se_n(z) = \sum_{r=0}^{\infty} B_{2r+1}^{(n)} \sinh (2r+1)z$$

$$= \tfrac{1}{2} \sum_{r=0}^{\infty} B_{2r+1}^{(n)}\, e^{(2r+1)} - \tfrac{1}{2} \sum_{r=0}^{\infty} B_{2r+1}^{(n)}\, e^{-(2r+1)z}$$

But $B_{-1} = -B_1$, so

$$Se_n(z) = \tfrac{1}{2} \sum_{r=-1}^{\infty} B_{2r+1}^{(n)}\, e^{(2r+1)z} - \tfrac{1}{2} \sum_{r=1}^{\infty} B_{2r+1}^{(n)}\, e^{-(2r+1)z}$$

$$= \tfrac{1}{2} \sum_{r=-1}^{\infty} B_{2r+1}^{(n)}\, e^{(2r+1)z} + O(e^{-3z}). \tag{12.1}$$

For $B_{2r+1}^{(n)}$ we substitute the factorial-series (3.1') with p=0; this gives, using $\tau = \left(\frac{k}{2}\right)^2$ (the factorial-series converges for $r > -2$)

$$Se_n(z) = \tfrac{1}{2} \sum_{r=-1}^{\infty} \frac{(-1)^r (k/2)^{2r}\, e^{(2r+1)z}}{(r+1)!} \sum_{\nu=0}^{\infty} \frac{\gamma_{\nu,0}^{(n)}}{(r+\nu)!} + O(e^{-3z})$$

$$= \frac{1}{k} \sum_{\nu=0}^{\infty} \gamma_{\nu,0}^{(n)} \sum_{r=-1}^{\infty} \frac{(-1)^2 \left(\frac{ke^z}{2}\right)^{2r+1}}{(r+1)!\,(r+\nu)!} + O(e^{-3z})$$

$$= \frac{1}{k} \gamma_{0,0}^{(n)} \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{ke^z}{2}\right)^{2r+1}}{r!\,(r+1)!} - \frac{1}{k} \sum_{\nu=1}^{\infty} \gamma_{\nu,0}^{(n)} \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{ke^z}{2}\right)^{2r-1}}{r!\,(r+\nu-1)!} + O(e^{-3z}) \tag{12.2}$$

One can easily account for the change of the order of summation. In the second term of (12.2) we have ($\nu \geqslant 1$)

$$\sum_{r=0}^{\infty} \frac{(-1)^r (\frac{ke^z}{2})^{2r-1}}{r!(r+\nu-1)!} = (\frac{ke^z}{2})^{-\nu} J_{\nu-1}(ke^z) \qquad (12.3)$$

in which

$$J_{\nu-1}(ke^z) = \frac{(\frac{ke^z}{2})^{\nu-1}}{(\nu-\frac{3}{2})!\sqrt{\pi}} \int_0^{\pi} \cos(ke^z \cos\varphi)\sin^{2\nu-2}\varphi \, d\varphi .$$

So the second term of (12.2) becomes

$$\frac{2e^{-z}}{k^2\sqrt{\pi}} \sum_{\nu=1}^{\infty} \frac{\gamma_{\nu,0}^{(n)}}{(\nu-\frac{3}{2})!} \int_0^{\pi} \cos(ke^z \cos\varphi)\sin^{2\nu-2}\varphi \, d\varphi$$

But

$$\gamma_{\nu,0}^{(n)} = O\,(\nu-3)!$$

and therefore

$$\frac{\gamma_{\nu,0}^{(n)}}{(\nu-\frac{3}{2})!} = O(\nu^{-3/2})$$

and finally

$$\left|\int_0^{\pi} \cos(ke^z \cos\varphi)\sin^{2\nu-2}\varphi \, d\varphi\right| < \pi ,$$

so the second term is $O(e^{-z})$.

Thus we find

$$Se_n(z) = \frac{1}{k}\gamma_{0,0}^{(n)} J_1(ke^z) + O(e^{-z}) \qquad (12.4)$$

and consequently

$$z\, Se_n(z) = \frac{1}{k}\gamma_{0,0}^{(n)} \left\{\log\frac{ke^z}{2} - \log\frac{k}{2}\right\} J_1(ke^z) + O(e^{-z}z). \qquad (12.5)$$

13. Now we have

$$Ge_n(z) = G_n(z) - zSe_n(z). \qquad (13.1)$$

with $G_n(z) = \sum_{r=0}^{\infty} g_{2r+1}^{(n)} \cosh(2r+1)z$

$$= \frac{1}{2}\sum_{r=0}^{\infty} g_{2r+1}^{(n)} e^{(2r+1)z} + \frac{1}{2}\sum_{r=0}^{\infty} g_{2r+1}^{(n)} e^{-(2r+1)z}$$

but $g_{-1} = g_1$, so

$$G_n(z) = \frac{1}{2}\sum_{r=-1}^{\infty} g_{2r+1}^{(n)} e^{(2r+1)z} + \frac{1}{2}\sum_{r=1}^{\infty} g_{2r+1}^{(n)} e^{-(2r+1)z} =$$

$$= \tfrac{1}{2}\sum_{r=-1}^{\infty} g_{2r+1}^{(n)} \; e^{(2r+1)z} + O(e^{-3z})$$

$$= \tfrac{1}{2}\sum_{r=-1}^{\infty} T_{2r+1}^{(n)} \; e^{(2r+1)z} + \frac{\vartheta}{2}\sum_{r=-1}^{\infty} B_{2r+1}^{(n)} \; e^{(2r+1)z} + O(e^{-3z}) \qquad (13.2)$$

The term $\frac{\vartheta}{2}\sum_{r=1}^{\infty} B_{2r+1}^{(n)} \; e^{(2r+1)z}$ has already been discussed in section 12 and yields

$$\frac{\vartheta}{2}\sum_{r=1}^{\infty} B_{2r+1}^{(n)} \; e^{(2r+1)z} = \frac{\vartheta}{k}\, \gamma_{0,0}^{(n)} \, J_1(ke^{z}) + O(e^{-z}). \qquad (13.3)$$

We deal in the same way with the term $\frac{1}{2}\sum_{r=-1}^{\infty} T_{2r+1}^{(n)} \; e^{(2r+1)z}$.

Inserting the factorial-series for the $T_{2r+1}^{(n)}$ with $p = 0$ we get

$$\tfrac{1}{2}\sum_{r=-1}^{\infty} T_{2r+1}^{(n)} \; e^{(2r+1)z} = \tfrac{1}{2}\sum_{r=-1}^{\infty} (-1)^r \frac{(\frac{k}{2})^{2r} \; e^{(2r+1)z}}{(r+1)!} \times$$

$$\sum_{\nu=0}^{\infty} \frac{\gamma_{\nu,0}^{(n)}}{(r+\nu)!} \; \frac{\Psi(r+1)+\Psi(r+\nu)}{2} = \frac{1}{k}\sum_{\nu=0}^{\infty} \gamma_{\nu,0}^{(n)} \sum_{r=-1}^{\infty} \frac{(-1)^r(\frac{ke^z}{2})^{2r+1}}{(r+1)!\,(r+\nu)!}$$

$$\frac{\Psi(r+1)+\Psi(r+\nu)}{2} = \frac{1}{k}\,\gamma_{0,0}^{(n)} \sum_{r=0}^{\infty} \frac{(-1)^r(\frac{ke^z}{2})^{2r+1}}{r!\,(r+1)!} \; \frac{\Psi(r)+\Psi(r+1)}{2}$$

$$- \frac{1}{k}\sum_{\nu=1}^{\infty} \gamma_{\nu,0}^{(n)} \sum_{r=0}^{\infty} \frac{(-1)^r(\frac{ke^z}{2})^{2r-1}}{r!\,(r+\nu-1)!} \; \frac{\Psi(r)+\Psi(r+\nu-1)}{2} \qquad (13.4)$$

In order to obtain an estimation of the second term of (13.4) in the same way as we did for the second term of (12.2) we first consider

$$a_r = \frac{(2r)!}{4^r(r+\alpha)!\,(r+\alpha+\nu-1)!} \qquad (13.5)$$

Using the duplication theorem of the factorial-function we get

$$a_r = \frac{1}{\sqrt{\pi}} \; \frac{r!\;(r-\frac{1}{2})!}{(r+\alpha)!\,(r+\alpha+\nu-1)!}$$

$$= \frac{1}{\sqrt{\pi}} \; \frac{1}{(\alpha+\nu-2)!\,(\alpha-\frac{1}{2})!} \; B(r+1,\alpha+\nu-1)\; B(r+\tfrac{1}{2},\alpha+\tfrac{1}{2})$$

$$= \frac{1}{\sqrt{\pi}} \; \frac{1}{(\alpha+\nu-2)!\,(\alpha-\frac{1}{2})!} \int_0^1 t^r(1-t)^{\alpha+\nu-2}dt.\int_0^1 u^{r-\frac{1}{2}}(1-u)^{\alpha-\frac{3}{2}}\,du$$

$$= \frac{1}{\sqrt{\pi}} \; \frac{1}{(\alpha+\nu-2)!\,(\alpha-\frac{1}{2})!} \int_0^1\!\!\int_0^1 (ut)^r(1-t)^{\alpha+\nu-2}\; u^{-\frac{1}{2}}\;(1-u)^{\alpha-\frac{1}{2}}\,du\,dt$$

$$= \frac{1}{\sqrt{\pi}} \; \frac{1}{(\alpha+\nu-2)!\,(\alpha-\frac{1}{2})!} \int_0^1 v^r\,d\,v \int_v^1 u^{\frac{3}{2}}\;(1-u)^{\alpha-\frac{1}{2}}\;(1-\frac{v}{u})^{\alpha+\nu-2}\,du$$

or if $w = \frac{v}{u}$

$$a = \frac{1}{\sqrt{\pi}} \frac{1}{(\alpha+\nu-2)!(\alpha-\frac{1}{2})!} \int_0^1 v^{r-\frac{1}{2}}\, dv \int_v^1 w^{-\alpha}(w-v)^{\alpha-\frac{1}{2}}(1-w)^{\alpha+\nu-2}\, dw. \tag{13.6}$$

Differentiating with respect to α and taking $\alpha = 0$, we get

$$\left(\frac{\partial a_r}{\partial\alpha}\right)_{\alpha=0} = -\frac{(2r)!}{4^r\, r!(r+\nu-1)!}\left[\psi(r)+\psi(r+\nu-1)\right]$$

$$= -\frac{1}{\sqrt{\pi}} \frac{\psi(\nu-2)+\psi(-\frac{1}{2})}{(\nu-2)!(-\frac{1}{2})!} \int_0^1 v^{r-\frac{1}{2}}\, dv \int_v^1 (w-v)^{-\frac{1}{2}}(1-w)^{\nu-2}\, dw$$

$$+ \frac{1}{\sqrt{\pi}} \frac{1}{(\nu-2)!(-\frac{1}{2})!} \int_0^1 v^{r-\frac{1}{2}}\, dv \int_v^1 (w-v)^{-\frac{1}{2}}(1-w)^{\nu-2}$$

$$\left[-\log w+\log(w-v)+\log(1-w)\right] dw \tag{13.7}$$

Now put

$$f_\nu(\alpha,v) = \frac{1}{(\alpha+\nu-2)!(\alpha-\frac{1}{2})!} \int_v^1 (w-v)^{\alpha-\frac{1}{2}}(1-w)^{\alpha+\nu-2}\, dw$$

$$= \frac{1}{(\alpha+\nu-2)!(\alpha-\frac{1}{2})!} \int_0^{1-v} w^{\alpha-\frac{1}{2}}(1-v-w)^{\alpha+\nu-2}\, dw$$

$$= \frac{(1-v)^{2\alpha+\nu-3/2}}{(\alpha+\nu-1)!(\alpha-\frac{1}{2})!} \frac{(\alpha-\frac{1}{2})!(\alpha+\nu-1)!}{(2\alpha+\nu-3/2)!}$$

$$= \frac{(1-v)^{2\alpha+\nu-\frac{3}{2}}}{(2\alpha+\nu-\frac{3}{2})!}$$

then

$$\left\{\frac{\partial f_\nu(\alpha,v)}{\partial\alpha}\right\}_{\alpha=0} = -\frac{\psi(\nu-2)+\psi(-\frac{1}{2})}{(\nu-2)!(-\frac{1}{2})!} \int_v^1 (w-v)^{-\frac{1}{2}}(1-w)^{\nu-2}\, dw$$

$$+ \frac{1}{(\nu-2)!(-\frac{1}{2})!} \int_v^1 (w-v)^{-\frac{1}{2}}(1-w)^{\nu-2}$$

$$\left[\log(w-v)+\log(1-w)\right] dw$$

$$= \frac{(1-v)^{\nu-\frac{3}{2}}}{(\nu-\frac{3}{2})!}\left[2\log(1-v)-2\psi(\nu-\tfrac{1}{2})\right]$$

and therefore

$$\left(\frac{\partial a_r}{\partial\alpha}\right)_{\alpha=0} = \frac{1}{\sqrt{\pi}} \int_0^1 v^{r-\frac{1}{2}}\, dv \left[\frac{2\log(1-v)-2\psi(\nu-\frac{1}{2})(1-v)^{\nu-\frac{3}{2}}}{(\nu-\frac{3}{2})!} - \int_v^1 (w-v)^{-\frac{1}{2}}(1-w)^{\nu-2}\log w\, dw\right]$$

$$= -\frac{(2r)!}{4^r r!(r+\nu-1)!}\left[\psi(r)+\psi(r+\nu-1)\right] \tag{13.8}$$

So
$$\sum_{r=0}^{\infty} \frac{(-1)^r (\frac{ke^z}{2})^{2r-1}}{r!(r+\nu-1)!} \quad \frac{\psi(r)+\psi(r+\nu-1)}{2}$$
$$= -\frac{1}{2\sqrt{\pi}} \sum_{r=0}^{\infty} \frac{(-1)^2}{(2r)!} \quad 4^r (\frac{ke^z}{2})^{2r-1} \quad (\frac{\partial a_r}{\partial \alpha})_{\alpha=0}$$
$$= -\frac{1}{\sqrt{\pi}} \int_0^1 \sum_{r=0}^{\infty} \frac{(-1)^r \; v^{r-\frac{1}{2}} (ke^z)^{2r-1}}{(2r)!} \, dv \; x$$
$$\left[\frac{2 \log(1-v) - 2\psi(\nu-\frac{1}{2})}{(\nu-\frac{3}{2})!} (1-v)^{\nu-3/2} - \int_v^1 (w-v)^{-\frac{1}{2}} (1-w)^{\nu-2} \log w \, dw \right].$$

So if $\xi^2 = v$
$$\sum_{r=0}^{\infty} \frac{(-1)^r (\frac{ke^z}{2})^{2r-1}}{r!(r+\nu-1)!} \quad \frac{\psi(r)+\psi(r+\nu-1)}{2} =$$
$$-\frac{2}{\sqrt{\pi}} \frac{1}{ke^z} \int_0^1 \cos(k\xi e^z) \frac{2 \log(1-\xi^2) - 2\psi(\nu-\frac{1}{2})}{(\nu-\frac{3}{2})!} (1-\xi^2)^{\nu-3/2} d\xi$$
$$+ \frac{2}{\sqrt{\pi}} \frac{1}{ke^z} \int_0^1 \cos(k\xi e^z) d\xi \int_{\xi^2}^1 (w-\xi^2)^{-\frac{1}{2}} (1-w)^{\nu-2} \log w \, dw.$$

The first integral is in absolute value certainly smaller than
$$\frac{1}{(\nu-\frac{3}{2})!} \int_0^1 \left[2\psi(\nu-\tfrac{1}{2}) - 2 \log(1-\xi^2) \right] (1-\xi^2)^{\nu-\frac{3}{2}} d\xi$$
$$< C \frac{\psi(\nu-\frac{1}{2})}{(\nu-\frac{3}{2})!}$$

and in the second we have if $\nu > 1$
$$\left| \int_{\xi^2}^1 (w-\xi^2)^{-\frac{1}{2}} (1-w)^{\nu-z} \log w \, dw \right|$$
$$< -2 \log \xi \int_{\xi^2}^1 (w-\xi^2)^{-\frac{1}{2}} (1-w)^{\nu-2} d\xi$$
$$= -C^* \frac{\log \xi \, (1-\xi^2)^{\nu-\frac{3}{2}}}{(\nu-\frac{3}{2})!}$$

This relation can be made true for $\nu = 1$ too by a small change of the value of C^*. So we finally find
$$\left| \sum_{r=0}^{\infty} \frac{(-1)^r (\frac{ke^z}{2})^{2r-1}}{r!(r+ -1)!} \quad \frac{\psi(r)+\psi(r+\nu-1)}{2} \right| < C^{**} e^{-z} \quad \frac{\psi(\nu-\frac{1}{2})}{(\nu-\frac{3}{2})!}$$

in which C^{**} can be chosen independent of ν and z.

So in the second term of (13.4) the general term of the sum with respect to ν is at most of the order $e^{-z}\nu^{-\frac{3}{2}}\log\nu$, therefore the term itself is at most of the order e^{-z} in z.

Then finally we have

$$\sum_{r=0}^{\infty}\frac{(-1)^r\left(\frac{ke^z}{2}\right)^{2r+1}}{r!(r+1)!}\;\frac{\psi(r)+\psi(r+1)}{2} = -\frac{\pi}{2}\,Y_1(ke^z) + \log\frac{ke^z}{2}\,J_1(ke^z) + \frac{1}{ke^z}$$

so we find

$$\frac{1}{2}\sum_{r=-1}^{\infty} T_{2r+1}^{(n)}\,e^{(2r+1)z} = -\frac{-\pi\gamma_{0,0}^{(n)}}{2k}\,Y_1(ke^z) + \frac{\gamma_{0,0}^{(n)}}{k}\log\left(\frac{ke^z}{2}\right)J_1(ke^z) + O(e^{-z}) \qquad (13.8)$$

Using (13.8), (13.3), (13.2), (13.1) and (12.5) we get

$$Ge_n(z) = -\frac{\pi}{2}\,\frac{\gamma_{0,0}^{(n)}}{k}\,Y_1(ke^z) + \frac{\gamma_{0,0}^{(n)}}{k}\left\{\vartheta + \log\frac{k}{2}\right\}J_1(ke^z) + O(e^{-z}) \qquad (13.9)$$

Now in (11.2) we have

$$Ne_n^{(2)}(z) = a\,Se_n(z) + b\,Ge_n(z) =$$

$$= \frac{\gamma_{0,0}^{(n)}}{k}\left[(a + b\vartheta + b\log\frac{k}{2})J_1(ke^z) - \frac{\pi}{2}\,b\,Y_1(ke^z)\right] + O(e^{-z})$$

So, in order that $Ne_n^{(2)}(z) \sim A\,H_1^{(2)}(ke^z)$ we must have

$$\frac{a + b\vartheta + b\log\frac{k}{2}}{\frac{\pi}{2}\,b} = -i$$

$$\frac{a}{b} = -\vartheta - \log\frac{k}{2} - i\,\frac{\pi}{2} \qquad (13.10)$$

14. Equally we find for even values of n

$$Se_n(z) = \frac{1}{k}\gamma_{0,0}^{(n)}\,J_1(Ke^z) + O(e^{-z}) \qquad (14.1)$$

and consequently

$$z\quad e_n(z) = \frac{1}{k}\gamma_{0,0}^{(n)}\left\{\log\frac{ke^z}{2} - \log\frac{k}{2}\right\}J_1(ke^z) + O(e^{-z}\,z) \qquad (14.2)$$

Also we have

$$Ge_n(z) = -\frac{\pi}{2}\,\frac{\gamma_{0,0}^{(n)}}{k}\,Y_1(ke^z) + \frac{\gamma_{0,0}^{(n)}}{k}\left\{\vartheta + \log\frac{k}{2}\right\}J_1(ke^z) + O(e^{-z}) \qquad (14.3)$$

Substitution of (14.1) and (14.3) in (11.2) gives:

$$Ne_n^{(2)}(z) = a\ Se_n(z) + b\ Ge_n(z) =$$

$$= \frac{\delta_{0,0}^{(n)}}{k}\left[(a+b\gamma+b\ \log \tfrac{k}{2})J_1(ke^z) - \tfrac{\pi}{2}\ b\ Y_1(ke^z)\right] + O(e^{-z})$$

So in order that $Ne^{(2)}(z) \sim A\ H_1^{(2)}(ke^z)$ we must have again the formula (13.10)

$$\frac{a}{b} = -\gamma - \log \frac{k}{2} - \frac{\pi i}{2} \qquad (14.4).$$